COMPLEX GENERATED BY VARIATIONAL DERIVATIVES. LAGRANGIAN FORMALISM OF INFINITE ORDER AND A GENERALIZED STOKES' FORMULA.

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1. The main construction. Let $M^{n|m}$ be a smooth (super)manifold. Consider the Lagrangians of $r|s$-dimensional paths, i.e. the maps $\Gamma : I^{r|s} \longrightarrow M^{n|m}$, where $I^{r|s} := I^r \times \mathbb{R}^{0|s}$ denotes the $r|s$-cube. Denote the space of these Lagrangians by $\Phi^{r|s}$. The corresponding functionals of action are of the form

$$S[\Gamma] = \int_{I^{r|s}} Dt\, L(x(t), \dot{x}(t), \dots).$$

Here and below we omit inessential common signs before the integrals. We assume that the Lagrangians may depend on derivatives of arbitrary (finite) order. If a Lagrangian depends on the derivatives of order $\leqslant k$, then it is well known that the variation of the action in general includes the derivatives of order $\leqslant 2k$. Here is the formula for the variation (cf. [1, p. 692]):

$$\delta S = \int_{I^{r|s}} Dt\, \delta x^A(t)\, \frac{\delta L}{\delta x^A}(x(t), \dot{x}(t), \dots),$$

where we introduced the notation

$$\frac{\delta L}{\delta x^A} := \sum_{l=0}^{\infty} (-1)^l (-1)^{\tilde{A}(\tilde{F}_1 + \tilde{F}_l)} \frac{\partial^l}{\partial t^{F_1} \dots \partial t^{F_l}} \frac{\partial L}{\partial x^A_{,F_1 \dots F_l}}.$$

(Mind that for the Lagrangians of arbitrary order the "total derivatives" w.r.t. the time variables are well-defined.) For each Lagrangian L we define a new Lagrangian $đL$, which determines the action for the paths of dimension advanced by one:

$$đL := \dot{x}^A_{r+1} \frac{\delta L}{\delta x^A}. \tag{1}$$

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Theorem 1. *The definition $đ$ is coordinate independent. The identity $đ^2 = 0$ holds and the operation $đ$ makes the space of all Lagrangians a cochain complex:*

$$\ldots \longrightarrow \Phi^{r|s} \xrightarrow{đ} \Phi^{r+1|s} \xrightarrow{đ} \Phi^{r+2|s} \xrightarrow{đ} \ldots \tag{2}$$

Proof. The variational derivatives transform under change of coordinates as the components of a covector. This implies coordinate independence of $đ$. Consider the square of $đ$. A straightforward calculation of $đđL$ in the general case is unavailable, since it would require the explicit expansion of all multiple total derivatives in the formula (1). For the Lagrangians of first order the formal expression for $đ^2$ contains 36 terms ! They do cancel, indeed, but checking this fact (the author has done it) is far from pleasant. Let us prove the identity $đ^2 = 0$ geometrically. It is convenient to introduce "weighted" paths and consider functionals of the form

$$S[\Gamma, g] = \int_{I^{r|s}} Dt\, g(t) L(x(t), \dot{x}(t), \ldots),$$

where the function g is compactly supported in the cube $I^{r|s}$. The following lemma holds.

Lemma. *For a variation of the map Γ*

$$\delta S = 0 \quad \textit{for} \quad \forall g\, \forall \delta x(t) : \operatorname{Supp} \delta x \cap \operatorname{Supp} dg = \varnothing \Longleftrightarrow \frac{\delta L}{\delta x^A}(x(t), \dot{x}(t), \ldots) = 0.$$

Proof. For the variations δx^A subjected to the above restriction it is not difficult to check that $\delta S = \int Dt\, g(t) \delta x^A\, \delta L/\delta x^A$. Suppose $\delta S = 0$. Take g supported inside a small ball, identically 1 near its center, and take $\delta x^A(t)$ as a product of a delta-like function in the even variables by an arbitrary function of the odd variables. Passing to the limit and making use of the nondegeneracy of the Berezin integral, we obtain the equality $(\delta L/\delta x^A)(t_0, \tau) \equiv 0$, where $t_0 \in I^r$ is arbitrary (the ball's center). $\square$

Consider the action S^* that corresponds to the Lagrangian $đL$. By definition of the variational derivative,

$$\frac{\partial x^A}{\partial t^{r+1}} \frac{\delta L}{\delta x^A}(x(t), \ldots) = \frac{\partial}{\partial t^{r+1}}(L) + \text{ derivatives w.r.t. } t^F, F \neq r+1.$$

Thus, the coefficient of $g^*(t^*)$ in the integral is a "total divergence". So, S^* can be rewritten as the integral of $\pm(\partial g^*/\partial t^{F^*}) h^{F^*}(t^*)$ (analog of the integral over boundary). Then for the variation $\delta x^A(t^*)$ satisfying the condition $\operatorname{Supp} \delta x \cap dg^* = \varnothing$, as it is easy to see, the equation $\delta S^* = 0$ always holds, since the support of δh^{F^*} is less or equal to the

support of δx^A. Applying Lemma , we obtain $\delta\, \bar{d}L/\delta x^A = 0$ identically, and thus $\bar{d}\bar{d}L = 0$. $\square$

The cohomology of the cochain complex (2) is an invariant of a smooth manifold M.

2. Generalized Stokes' Formula. Suppose we have a family of D-paths Γ_s (where $s \in [0,1]$) that is constant outside of a compact in I^D, $D = r|s$. We consider it as a path Γ^*, of dimension $D+1$. As an immediate corollary of the above consideration we obtain the following

Theorem 2. *For any Lagrangian*

$$S[\Gamma_1] - S[\Gamma_0] = \int_{\Gamma^*} \bar{d}L. \tag{3}$$

If the Lagrangian is such that the action is independent of parametrization (with an appropriate restriction on orientation, see [2]), then it is possible to integrate it over singular manifolds with boundary, and a complete analog of the Stokes' Formula is valid. We see that it is no exclusive property of the differential forms. The formula (3) provides explanation of the identity $\bar{d}^2 = 0$ as a manifestation of the general fact that the "boundary of boundary is zero".

3. Application to supermanifold forms. Definition (1) for the first time was suggested by the author in the following context [3]: one had to define the right analog of the de Rham complex for a supermanifold that is not an even manifold. It was suggested to consider as forms the "covariant" Lagrangians of the first order, satisfying an extra condition that provided the independence of variation of the accelerations (see [3, 2]). For the even case it was proved that the de Rham–Cartan theory was reproduced. In general situation the theory is nontrivial. The indicated condition is equivalent to a system of PDE [2, p. 58], for which the odd-odd part gives the equations well known in the connection with Radon-like integral transforms [4, 5]. Let us introduce a filtration of Lagrangians by the order of derivatives. Then our system of equations receive a "homological" interpretation as the condition for the elements under consideration to keep their filtration under the action of the differential $\bar{d}$. The higher analogs of this system are obvious in such interpretation. The results of the present note provide substantial simplification of the proofs from [2]. Recently a "second half" of the complex of supermanifold forms was constructed, with the use of the "dual" language [6]. I hope that the combination of the methods of [6] with those of the current paper will lead to new advances.

4. Application to the inverse problem of the calculus of variations. According to Theorem 1, in order to check, if some set

of functions f_A coincide with variational derivatives of a Lagrangian of $r|s$-path, it is necessary to calculate $\bar{d}(\dot{x}^A_{r+1} f_A)$ and check that it is zero. (Note that the cross-sections of bundles also can be treated as "surfaces" or "paths".) The inverse problem of the calculus of variations has been studied in [7, 8] and others. It would be interesting to compare the approach of these papers with our complex (2).

As a conclusion I wish to notice that the constructions of differentials known in homology theory fall into two or three main types (combinatorial differentials and E. Cartan or Koszul type differentials). The "variational" differentials, studied in [3, 2, 6] and in the present note, are likely to supply yet another type.

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